Brownian Carnot engine
Supplementary Information

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S1. EXPERIMENTAL TESTS ON STOCHASTIC EFFICIENCY

The fluctuations of the efficiency of heat engines have been been characterized in the framework of stochastic thermodynamics [1–5]. Universal properties of the probability density function (PDF) [4] of the efficiency and of its large deviation function (LDF) [2, 3, 6] have recently been established from the application of fluctuation theorems to mesoscopic engines. The experimental verification of the majority of these results is however still lacking.

Two of the major theoretical predictions of the efficiency PDF, namely the bimodality of the histogram near the maximum power output [4] and the power-law tails [7], have been tested experimentally in Fig. 3 in the Main Text. In what follows, we review some of the main theoretical predictions for the LDF of the stochastic efficiency and show the experimental test of some of these features in our Carnot micro engine.

In the limit of large observation times, the efficiency distribution can be characterized by its LDF. The LDF of the efficiency fluctuations, $J_\tau(\eta)$, describes the asymptotic behaviour of the efficiency PDF when the efficiency is calculated summing over a large number of cycles

$$\rho(\eta_\tau^{(i)}) \simeq e^{-iJ_\tau(\eta)} \quad , \quad \text{for} \quad i\to\infty \quad , \tag{S1}$$

where the subindex τ in $J_\tau(\eta)$ indicates the duration of the cycle of the engine. Here $\rho(\eta_\tau^{(i)})$ is the PDF of the efficiency obtained as the ratio of the cumulative sum of the work W over i cycles over the total heat absorbed ($Q = Q_1+Q_2+Q_3$) summed over i cycles. From Eq. (S1), the LDF of the efficiency can be estimated as

$$J_\tau(\eta) \simeq -\lim_{i\to\infty}\frac{1}{i}\ln\rho(\eta_\tau^{(i)}) \quad . \tag{S2}$$

Introducing the observation time $\tau_{\rm obs} = i\tau$, we can also define the LDF in units of inverse time,

$$\mathcal{J}_\tau(\eta) \simeq -\lim_{\tau_{\rm obs}\to\infty}\frac{1}{\tau_{\rm obs}}\ln\rho(\eta_\tau^{(\tau_{\rm obs})}) \quad . \tag{S3}$$

In time-symmetric cycles the efficiency LDF has been found to attain its global maximum at the Carnot value η_C [2]. In other words, Carnot efficiency is the least likely efficiency $\eta_{\rm min}$ in time-symmetric cycles: $\eta_{\rm min}^{\rm sym} = \eta_C$. For time-asymmetric cycles like our Carnot engine, however, an off-Carnot maximum of the LDF has been predicted theoretically, $\eta_{\rm min}^{\rm asym} \neq \eta_C$ [3, 6]. In such a case, the stochastic entropy production $\Delta S_{\rm tot}$ vanishes when averaged over ensembles of trajectories whose efficiency equals to η_C and also when averaged over ensembles of trajectories whose efficiency equals to $\eta_{\rm min}$ [3]: $\langle\Delta S_{\rm tot}\rangle_\eta = 0$ for $\eta = \eta_C, \eta_{\rm min}$.

The first theoretical results on stochastic efficiency have been obtained under the assumption of Gaussian work and heat fluctuations [4] or when the joint distribution of heat and work is smooth around zero [3]. Figure S1 shows that the experimental distributions of the extracted work and the total absorbed heat can be approximatively described as Gaussians. Note that the work distributions at an observation time equivalent to 10 cycles fit better to a Gaussian distribution than that heat summed over 10 cycles of the engine. Both work and heat rate functions $\ln[\rho_\tau^{(\tau_{\rm obs})}(W/\tau_{\rm obs})]/\tau_{\rm obs}$ and $\ln[\rho_\tau^{(\tau_{\rm obs})}(Q/\tau_{\rm obs})]/\tau_{\rm obs}$ collapse to a universal curve for observation times equal or larger than ~ 20 cycles (cf. Fig. 2 in [3]).

The convergence of work and heat PDFs with $\tau_{\rm obs}$ at low observation times (~ 20 cycles) suggests that the efficiency LDF could be accurately estimated by the value of the efficiency rate function $-\ln[\rho_\tau^{(\tau_{\rm obs})}(\eta)]\,/\,\tau_{\rm obs}$ for $\tau_{\rm obs}$ of the order of the duration of several tenths of cycles. Figure S2 shows the value of $\ln[\rho_\tau^{(\tau_{\rm obs})}(\eta)]\,/\,\tau_{\rm obs}$ for different values of $\tau_{\rm obs}$. At observation times corresponding to 10, 20 and 30 cycles our engine attains a minimum of the efficiency PDF at an off-Carnot value, $\eta_{\rm min} \simeq 2.5\,\eta_C$ as predicted by the theory of stochastic efficiency for asymmetric cycles [3]. Using an extrapolation technique described in Sec. S2 we obtain an estimation of the efficiency LDF that lies between the efficiency rate function

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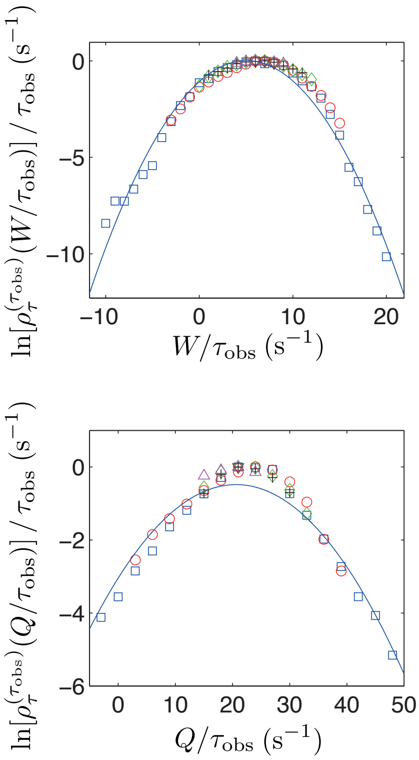

FIG. S1: **Work and heat fluctuations for the Carnot cycle of duration** $\tau = 40$ms. Top: Work rate function $\ln[\rho_\tau^{\tau_{\rm obs}}(W/\tau_{\rm obs})]/\tau_{\rm obs}$ as a function of the work scaled by the observation time. The data is obtained for different observation times $\tau_{\rm obs}$ corresponding to 10 cycles (blue squares), 20 cycles (red circles), 40 cycles (green diamonds), 50 cycles (black "+"), 100 cycles (magenta up triangles) and 200 cycles (brown down triangles). Bottom: Rate function of the total absorbed heat measured at the same observation times. Both heat and work are measured in units of kT_c, with $T_c = 300\,$K and k Boltzmann's constant. All the distributions are normalized to their maximum value. The solid lines are fits of the 10-cycle distributions to a Gaussian distribution.

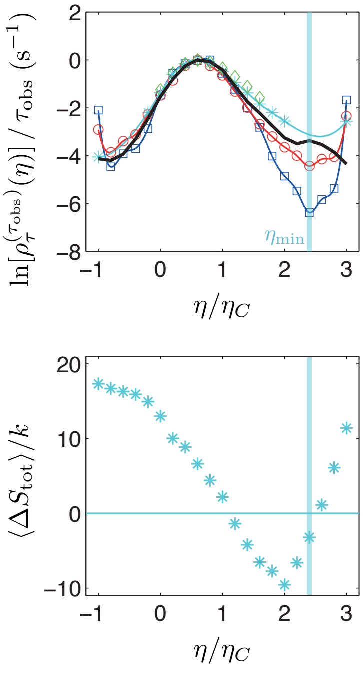

FIG. S2: **Efficiency large deviation function and mean entropy production for the Carnot cycle of duration** $\tau = 40$ms. Top: Rate function of the efficiency (normalized to the maximum value) for different observation times corresponding to 10 cycles (blue squares), 20 cycles (red circles), 30 cycles (cyan stars) and 40 cycles (green diamonds). The data is obtained using a regular binning from $-\eta_C$ to $3\eta_C$ with bin size $0.2\eta_C$. The black line is the efficiency LDF calculated using the method described in Sec. S2 and the solid lines connecting the symbols are obtained with a spline interpolation. Bottom: Mean entropy production as a function of the efficiency for an observation time of 30 cycles. Mean entropy production vanishes at $\eta \simeq \eta_C$ and near $\eta \simeq \eta_{\rm min}$. Here, $\eta \simeq \eta_{\rm min}$ is estimated from the minimum of the efficiency PDFs shown in the top figure (vertical cyan line).

calculated for 20 and 30 cycles (black curve in Fig. S2). Note that the entropy production vanishes when averaged over cycles that perform an efficiency equal to $\eta_{\rm min}$, as does when averaged over cycles with efficiency equal to the Carnot value (Fig. S2 bottom, cf. Fig. 3 in [3]).

Figure S3 shows the distribution of the stochastic efficiency $\eta_{\rm loc}$ defined in [1] as the ratio between the work extracted and the heat absorbed in one cycle, $\eta_{\rm loc} = \eta_\tau^{(1)}$. Our experimental result confirms the theoretical prediction for the one-cycle efficiency distribution, which can be well described by a Cauchy distribution, as predicted for the case of engines with Gaussian heat and work fluctuations [4].

S2. ESTIMATION OF THE EFFICIENCY LDF FROM FINITE-TIME OBSERVATIONS

In experimental time series of a finite duration $\tau_{\rm exp}$, the statistics of $\rho(\eta_\tau^{(\tau_{\rm obs})})$ for a large observation time $\tau_{\rm obs}$ is limited. The estimation of the efficiency LDF using Eqs. (S2) and (S3) is therefore subject to possible statistic shortcomings in the long-time limit. We design an alternative estimator by extrapolating the rate function $-\ln[\rho(\eta_\tau^{(\tau_{\rm obs})})]/\tau_{\rm obs}$ to $\tau_{\rm obs} = \infty$ from the efficiency PDFs $\rho(\eta_\tau^{(\tau_{\rm obs})})$ for $\tau_{\rm obs}$ small, where the statistics is more robust. Empirically, we find the following finite-time correction for the LDF,

$$-\frac{1}{\tau_{\rm obs}}\ln\rho(\eta_\tau^{(\tau_{\rm obs})}) = \mathcal{J}_\tau(\eta) + \frac{B}{\tau_{\rm obs}} \quad , \tag{S4}$$

as shown in Fig. S4 for $\eta = \eta_C$ and $\eta = 2\eta_C$. As

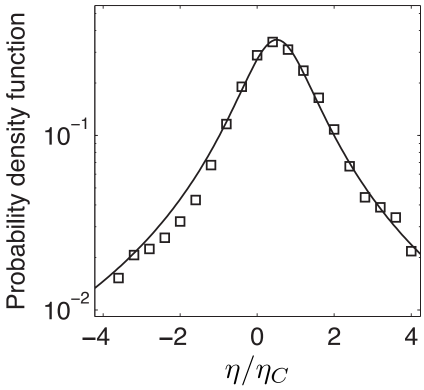

FIG. S3: **Distribution of stochastic efficiency $\eta_{\rm loc}$ for the Carnot cycle with $\tau = 40$ms.** Experimental value of the one-cycle efficiency distribution obtained with a regular binning of $0.2\eta_C$ (black squares) and fit to a Cauchy distribution (black line). The goodness of the fit is $R^2 = 0.998$.

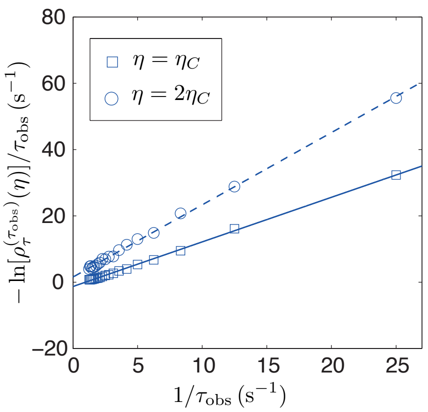

FIG. S4: **Estimation of the efficiency LDF from finite $\tau_{\rm obs}$ in the Carnot cycle with $\tau = 40$s.** Rate function of the efficiency PDF as a function of the inverse observation time for two different values of the efficiency corresponding to $\tau_{\rm obs}$ up to 20 cycles. The LDF is estimated as the y-intercept of the linear fit of the rate function vs the inverse observation time (solid and dashed lines). In the data shown here $\mathcal{J}_\tau(2\eta_C) > \mathcal{J}_\tau(\eta_C)$.

a result, $\mathcal{J}_\tau(\eta)$ can be estimated from the y-intercept of a linear fit of $-\frac{1}{\tau_{\rm obs}}\ln\rho(\eta_\tau^{(\tau_{\rm obs})})$ vs $1/\tau_{\rm obs}$. Figure S5 shows the value of the estimator of $\mathcal{J}_\tau(\eta)$ obtained using this method. In the extrapolation, we use the data of the PDFs $\rho_\tau(\eta^{\tau_{\rm obs}})$ for $\tau_{\rm obs}$ ranging from 1 cycle period to $\tau_{\rm max} = 5\tau$ (yellow crosses), 10τ (blue squares), 15τ (magenta stars) and 20τ (red circles). Our estimator of $\mathcal{J}_\tau(\eta)$ converges for $\tau_{\rm max} \sim 20\tau$ therefore confirming that the rate function estimation introduced in Sec. S1 is also an accurate estimator of the efficiency LDF.

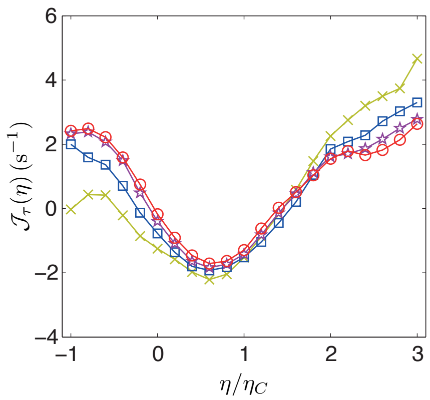

FIG. S5: **Efficiency LDF in the Carnot cycle of duration $\tau = 40$ms.** The data is obtained from linear extrapolation of $-\frac{1}{\tau_{\rm obs}}\ln\rho(\eta_\tau^{(\tau_{\rm obs})})$ vs $1/\tau_{\rm obs}$ using the data of efficiency PDFs with $\tau_{\rm obs}$ ranging from 1 cycle to different number of cycles: 5 cycles (yellow crosses), 10 cycles (blue squares), 15 cycles (magenta stars) and 20 cycles (red circles). The red curve corresponds with the solid black curve in Fig. S2. Solid lines are a guide to the eye.

[1] Sekimoto, K., Takagi, F. & Hondou, T. Carnot's cycle for small systems: Irreversibility and cost of operations. *Phys. Rev. E* **62**, 7759–7768 (2000).

[2] Verley, G., Esposito, M., Willaert, T. & den Broeck, C. V. The unlikely Carnot efficiency. *Nat. Commun.* **5**, 5721 (2014).

[3] Gingrich, T. R., Rotskoff, G. M., Vaikuntanathan, S. & Geissler, P. L. Efficiency and large deviations in time-asymmetric stochastic heat engines. *New J. Phys.* **16**, 102003 (2014).

[4] Polettini, M., Verley, G. & Esposito, M. Efficiency statistics at all times: Carnot limit at finite power. *Phys. Rev. Lett.* **114**, 050601 (2015).

[5] Rana, S., Pal, P., Saha, A. & Jayannavar, A. Single-particle stochastic heat engine. *Phys. Rev. E* **90**, 042146 (2014).

[6] Verley, G., Willaert, T., Van den Broeck, C. & Esposito, M. Universal theory of efficiency fluctuations. *Phys. Rev. E* **90**, 052145 (2014).

[7] Proesmans, K., Cleuren, B. & Van den Broeck, C. Stochastic efficiency for effusion as a thermal engine. *EPL (Europhys. Lett.)* **109**, 20004 (2015).